Expansions for the Logarithmic Kramers–Kronig Relations

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Abstract

The log transfer function of a time–invariant linear system which is causal and minimum phase satisfies relations of the Kramers–Kronig type which allow, for example, to calculate its real part U from its imaginary part V. A remarkably simple approximation to this relation of the form

$$U(w) \approx \frac{2}{\pi}\int_0^w V(x)\,d\log x \,+\, \alpha\,\frac{dV(w)}{d\,\log w} \,+\, const.$$

was proposed by Göhr and Schiller in 19?? for application in electrical impedance spectroscopy. In this note an expansion of U in terms of V will be established which may be considered as a mathematical justification for this approximation.

Keywords and Phrases: Approximation, causal system, Hardy function, Kramers–Kronig relations, minimum phase, series expansion, transfer function.

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1 Introduction: Kramers-Kronig relations

Consider a time–invariant linear system which transforms an incoming signal $f(t)$ into a signal $g(t)$ given by $g(t) = \int_{-\infty}^{\infty} k(t-s)\, f(s)\, ds$. Suppose the filter k is causal, i.e. $k(t) = 0$ for $t < 0$, and square–integrable. Then the transfer function $\hat{k}(\omega) = \int_0^\infty e^{i\omega t}\, k(t)\, dt$ is a *Hardy function* (cf. Dym and McKean (1972), sections 3.4, 3.5): it is analytic in the upper half plane $\mathbf{C}_+ = \{\omega \in \mathbf{C} : \ Im\, \omega > 0\}$ and satisfies

$$\sup_{b>0} \int_{-\infty}^{\infty} |\hat{k}(a+ib)|^2\, da < \infty. \tag{1.1}$$

The analyticity of the transfer function implies strong relations between its real and imaginary part even on the boundary of $\mathbf{C}_+$, the real line $\mathbf{R}$. These relations, known in system theory and physics as the Kramers–Kronig or dispersion relations, basically say that the real and imaginary parts of $\hat{k}$ are Hilbert transforms of each other. For real–valued k they may be written in the form

$$Im\,\hat{k}(w) = -\frac{2w}{\pi}\, PV\!\!\int_0^\infty \frac{Re\,\hat{k}(x)}{x^2 - w^2}\, dx, \qquad Re\,\hat{k}(w) = \frac{2}{\pi}\, PV\!\!\int_0^\infty \frac{x\ Im\,\hat{k}(x)}{x^2 - w^2}\, dx, \tag{1.2}$$

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with $PV\!\int$ denoting the Cauchy principal value and $w \in \mathbf{R}$.

If the transfer function also is "minimum phase", that is, has no root in $\mathbf{C}_+$, then the log transfer function, too, is analytic in $\mathbf{C}_+$ and should satisfy analogous relations. Let

$$U(\omega) = \log|\hat{k}(\omega)| = Re\,\log\hat{k}(\omega), \qquad V(\omega) = \arg\hat{k}(\omega) = Im\,\log\hat{k}(\omega). \tag{1.3}$$

The Kramers–Kronig type relations for the log transfer function $\log\hat{k} = U + iV$, to be called the *logarithmic Kramers–Kronig relations*, read as follows.

$$V(w) = -\frac{2w}{\pi} PV\!\int_0^\infty \frac{U(x)}{x^2 - w^2}\,dx, \tag{1.4}$$

$$U(w) - U(0) = \frac{2w^2}{\pi} PV\!\int_0^\infty \frac{V(x)}{x(x^2 - w^2)}\,dx. \tag{1.5}$$

The first relation formally is the same as in (1.2) but the second must be different, due to problems with the convergence of the relevant integral. The validity of the relations can be established under the following set of assumptions (A_q), which depend on an integer $q \geq 0$. The argument will be sketched in the appendix.

ASSUMPTIONS (A_q). k is a real-valued function on $\mathbf{R}$ vanishing for $t < 0$ and such that $\int_0^\infty t^q\,|k(t)|\,dt < \infty$. Its Fourier transform $\hat{k}$ has no zero in the closed upper half-plane $\overline{\mathbf{C}}_+$ and satisfies the following conditions.

$$\lim_{R\to\infty} R^{-(1+q)} \int_0^\pi |\log\hat{k}(Re^{i\varphi})|\,d\phi = 0; \tag{1.6}$$

$$\int_{-\infty}^\infty \frac{|\log\hat{k}(x)|}{1 + x^{2+q}}\,dx < \infty. \tag{1.7}$$

The focus in this paper will be on the relation (1.5). In section 2 we establish an expansion for the right–hand side of (1.5) which may be viewed as a mathematical justification for the approximation

$$U(w) \approx -\frac{2}{\pi}\int_0^w V(x)\,d\log x + \alpha\,\frac{dV(w)}{d\,\log w} + const. \tag{1.8}$$

Göhr and Schiller (19??) introduced this approximation in connection with electrical impedance spectroscopy on the basis of an important special case where it is exact, and of empirical evidence. In an ideal electrical device the impedance modulus – the real part of the log transfer function – should be related to the phase – the imaginary part – through (1.5). The approximation thus may be (and is) used for quality control purposes. Section 3 contains a brief discussion of the reverse relation (1.4).

2 An expansion for (1.5)

In the sequel the k-th derivative of $V(x)$ with respect to the variable $\log x$ evaluated at $t \in \mathbf{R}$ will be denoted by $V_k(t)$. It may be defined recursively starting with $V_1(t) =$

$\frac{dV(t)}{d\log t} = t\,V'(t)$ for $k = 1$. Notice that U and V are even and odd functions on $\mathbf{R}$, respectively. In particular, $V(0) = 0$. Therefore it suffices to consider the physically meaningful case of positive frequencies $w > 0$ only. Finally, $\zeta(s) = \sum_{n\geq 1} n^{-s}$ denotes the Riemann ζ function.

Proposition 2.1 *Assume (A_1), and suppose V is $n \geq 2$ times continuously differentiable and satisfies $\lim_{t\to 0} V(1/t)\, t^2 = 0$. Then for every $w > 0$*

$$\frac{\pi}{2}\,(U(w) - U(0)) = -\int_0^w V(x)\, d\log x \;+ \sum_{k=1,\, k\ odd}^{n-1} V_k(w)\;\zeta(k+1)\;2^{-k} \;+\; R_n(w)\,. \quad (2.1)$$

The remainder term is given by

$$R_n(w) \;=\; \int_0^1 (V_n(w/r) - (-1)^n V_n(wr))\,\sigma_{n-2}(r)\,\frac{dr}{r}, \qquad where \quad (2.2)$$

$$\sigma_k(r) \;=\; \int_0^r \frac{(\log\frac{r}{t})^k}{k!}\,(-\frac{1}{2}\log(1-t^2))\,\frac{dt}{t} \qquad (k \geq 0,\; 0 \leq r \leq 1). \quad (2.3)$$

The following estimates hold:

$$|R_n(w)| \;\leq\; 2^{-n}\,(1 + \frac{1}{2}\zeta(n))\;\sup_x |V_n(x)|\,, \quad (2.4)$$

$$\int_0^\infty |R_n(w)|\,\frac{dw}{w} \;\leq\; 2^{-n}\,(1 + \frac{1}{2}\zeta(n))\;\int_0^\infty |V_n(x)|\,\frac{dx}{x}\,. \quad (2.5)$$

PROOF. Let $n > 1$ and $w > 0$ be fixed. The proof is based on (1.5), which is valid under (A_1) (see Appendix), and an analysis of the expression

$$\begin{aligned}\Delta(w) \;&:=\; \frac{\pi}{2}\,(U(w) - U(0)) \;+\; \int_0^w V(x)\frac{dx}{x} \qquad (2.6)\\ &=\; w^2 \int_0^\infty \frac{V(x)}{x(x^2-w^2)}\,dx \;+\; \int_0^w V(x)\frac{dx}{x}\,.\end{aligned}$$

In (2.6) we have dropped the PV sign, and we will stick to this from now on. Changing variables and then breaking up the first integral into two we get

$$\begin{aligned}\Delta(w) \;&=\; \int_0^\infty \frac{V(wt)\,dt}{(t^2-1)\,t} \;+\; \int_0^1 V(wt)\frac{dt}{t} \qquad (2.7)\\ &=\; \int_0^1 V(wt)\left(1 + \frac{1}{t^2-1}\right)\frac{dt}{t} \;+\; \int_1^\infty \frac{V(wt)\,dt}{(t^2-1)\,t}\\ &=\; \int_0^1 V(wt)\,\frac{t}{t^2-1}\,dt \;+\; \int_0^1 \frac{V(w/s)\,s\,ds}{(s^{-2}-1)\,s^2}\\ &=\; \int_0^1 (V(w/t) - V(wt))\,\frac{t}{1-t^2}\,dt.\end{aligned}$$

This representation of Δ will now be used to derive the desired expansion. Observing

$$\int_w^{x_0} \cdots \int_w^{x_{n-1}} d\log x_n \cdots d\log x_1 = \frac{(\log\frac{x_0}{w})^n}{n!} \qquad (x_0 > 0) \quad (2.8)$$

(the integral is understood to extend over all $(x_1,\dots,x_n)$ such that $w \le x_n \le \dots \le x_1 \le x_0$ if $w < x_0$, or such that $w \ge x_n \ge \dots \ge x_1 \ge x_0$ if $w > x_0$) one easily verifies by induction that

$$V(x_0) - V(w) = \sum_{k=1}^{n-1} V_k(w)\,\frac{(\log\frac{x_0}{w})^k}{k!} + \int_w^{x_0} K_n(w,x_1)\,d\log x_1\,, \tag{2.9}$$

where

$$\begin{aligned} K_n(w,x_1) &= \int_w^{x_1}\cdots\int_w^{x_{n-1}} V_n(x_n)\,d\log x_n\cdots d\log x_2 \\ &= \int_w^{x_1} V_n(x_n)\,\frac{(\log\frac{x_1}{x_n})^{n-2}}{(n-2)!}\,d\log x_n\,. \end{aligned} \tag{2.10}$$

Therefore

$$\begin{aligned} \Delta(w) &= \int_0^1 (V(w/t)-V(w))\,\frac{t\,dt}{1-t^2} - \int_0^1 (V(wt)-V(w))\,\frac{t\,dt}{1-t^2} \\ &= \sum_{k=1}^{n-1} V_k(w)\,\frac{1}{k!}\int_0^1 \left((\log\frac{w/t}{w})^k - (\log\frac{wt}{w})^k\right)\frac{t\,dt}{1-t^2} + R_n(w) \\ &= \sum_{k=1,\ k\ \mathrm{odd}}^{n-1} V_k(w)\,\frac{-2}{k!}\int_0^1 (\log t)^k\,\frac{t\,dt}{1-t^2} + R_n(w) \end{aligned} \tag{2.11}$$

with a remainder term given by $R_n = R_n^+ - R_n^-$, where

$$R_n^+(w) = \int_0^1\int_w^{w/t} K_n(w,x_1)\,d\log x_1\,\frac{t\,dt}{1-t^2}\,, \tag{2.12}$$

$$R_n^-(w) = \int_0^1\int_w^{wt} K_n(w,x_1)\,d\log x_1\,\frac{t\,dt}{1-t^2}\,. \tag{2.13}$$

The coefficients $c_k = \frac{-2}{k!}\int_0^1 (\log t)^k\,\frac{t\,dt}{1-t^2}$ can be evaluated using the wellknown formula

$$\int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx = \Gamma(s)\,\zeta(s). \tag{2.14}$$

Substituting $t = e^{-x/2}$ we find

$$c_k = \frac{(-1/2)^k}{k!}\int_0^\infty \frac{x^k}{e^x-1}\,dx = (-1)^k\,2^{-k}\,\zeta(k+1).$$

Therefore, the expansion (2.1) is established up to the analysis of the remainder term. Consider R_n^+. Partial integration gives

$$R_n^+(w) = \int_0^1\int_w^{w/t} K_n(w,x_1)\,\frac{dx_1}{x_1}\,\frac{t\,dt}{1-t^2} = \int_0^1 K_n(w,w/t)\left(-\frac{1}{2}\,\log(1-t^2)\right)\frac{dt}{t}.$$

The boundary terms do vanish. This can be proved by going back to (2.9) – which allows us to rewrite the integral $\int_w^{w/t} K_n(w,x_1)\,\frac{dx_1}{x_1}$ as a linear combination of the terms

$V(w/t) - V(w)$ and $(\log(1/t))^k$ $(1 \le k < n)$ – and using the assumptions about V. With (2.10) we then obtain

$$\begin{aligned} R_n^+(w) &= \int_0^1 \int_w^{w/t} V_n(s) \frac{(\log \frac{w}{st})^{n-2}}{(n-2)!} \frac{ds}{s} \left(-\frac{1}{2} \log(1-t^2)\right) \frac{dt}{t} \\ &= \int_0^1 V_n(w/r) \int_0^r \frac{(\log \frac{r}{t})^{n-2}}{(n-2)!} \left(-\frac{1}{2} \log(1-t^2)\right) \frac{dt}{t} \frac{dr}{r} \\ &= \int_0^1 V_n(w/r)\, \sigma_{n-2}(r) \frac{dr}{r}. \end{aligned} \tag{2.15}$$

The other remainder term can be treated similarly. One obtains

$$R_n^-(w) = (-1)^n \int_0^1 V_n(wr)\, \sigma_{n-2}(r) \frac{dr}{r},$$

and together with (2.15) this proves (2.2).
We now estimate the total mass of the measure $\sigma_k(r)\,\frac{dr}{r}$.

$$\begin{aligned} \int_0^1 \sigma_k(r) \frac{dr}{r} &= \int_0^1 \int_t^1 \frac{(\log r - \log t)^k}{k!} d\log r \left(-\frac{1}{2} \log(1-t^2)\right) \frac{dt}{t} \\ &= \int_0^1 \frac{(-\log t)^{k+1}}{(k+1)!} \left(-\frac{1}{2} \log(1-t^2)\right) \frac{dt}{t} \\ &= \int_0^1 \frac{(-\frac{1}{2}\log s)^{k+1}}{(k+1)!} \left(-\frac{1}{2} \log(1-s)\right) \frac{ds}{2s}. \end{aligned}$$

Applying the inequality $-\log(1-s) \le s\,(1 + \frac{1}{2}\frac{s}{1-s})$, $0 \le s < 1$, and substituting $s = e^{-x}$ we obtain the desired estimate,

$$\begin{aligned} \int_0^1 \sigma_k(r) \frac{dr}{r} &\le 2^{-(k+3)} \int_0^1 \frac{(-\log s)^{k+1}}{(k+1)!} ds + 2^{-(k+4)} \int_0^1 \frac{(\log \frac{1}{s})^{k+1}}{(k+1)!} \frac{ds}{s^{-1}-1} \\ &\le 2^{-(k+3)} \int_0^\infty \frac{x^{k+1}}{(k+1)!} e^{-x}\, dx + 2^{-(k+4)} \int_0^\infty \frac{x^{k+1}}{(k+1)!} \frac{e^{-x}\, dx}{e^x - 1} \\ &\le 2^{-(k+3)} \left(1 + \frac{1}{2}\zeta(k+2)\right). \end{aligned} \tag{2.16}$$

The pointwise error bound (2.4) immediately follows from (2.2) and (2.16). The integral bound (2.5), finally, follows from

$$\begin{aligned} \int_0^\infty |R_n(w)| \frac{dw}{w} &\le \int_0^\infty \int_0^1 (|V_n(w/r)| + |V_n(wr)|)\, \sigma_{n-2}(r) \frac{dr}{r} \frac{dw}{w} \\ &= \int_0^\infty \int_w^\infty |V_n(s)|\, \sigma_{n-2}(w/s) \frac{ds}{s} \frac{dw}{w} + \int_0^\infty \int_0^w |V_n(s)|\, \sigma_{n-2}(s/w) \frac{ds}{s} \frac{dw}{w} \\ &= \int_0^\infty \int_0^s \sigma_{n-2}(w/s) \frac{dw}{w} |V_n(s)| \frac{ds}{s} + \int_0^\infty \int_s^\infty \sigma_{n-2}(s/w) \frac{dw}{w} |V_n(s)| \frac{ds}{s} \\ &= 2 \int_0^1 \sigma_{n-2}(u) \frac{du}{u} \int_0^\infty |V_n(s)| \frac{ds}{s}. \end{aligned}$$

Remark 2.1 The quality of an approximation obtained by ignoring the remainder term in (2.1) depends on the behaviour of the derivatives of V with respect to $\log w$. If these do not increase too fast with n, the exponential factor 2^{-n} will make the approximation effective for large enough n. For fixed n the remainder term is not negligible, however, and the numeric value of the coefficients $\zeta(k+1)\,2^{-k}$ should not be taken too seriously. Optimizing the coefficients, for fixed n, with respect to a suitable error criterion may produce better results in some cases.

Remark 2.2 Filters in the form of a power or an exponential law represent two simple examples of particular interest.
The transfer function corresponding to $k(t) = t^{\alpha-1}$, $t > 0$, is $\hat{k}(\omega) = \Gamma(\alpha)/(-i\omega)^{\alpha}$. Our assumptions are not satisfied in this case. Nevertheless, one may note that $\log \hat{k}(\omega) = \log\Gamma(\alpha) - \alpha\,\log|\omega| - i\alpha\,(\arg(\omega) - \frac{\pi}{2})$, so that for real w (where $\arg(w) = 0$ or $= \pi$ according to $w > 0$ or $w < 0$, respectively),

$$U(w) = \log\Gamma(\alpha) - \alpha\,\log|\omega|, \qquad V(w) = \alpha\,\frac{\pi}{2}\,\mathrm{sign}(w).$$

Apart from the jump at $w = 0$ the phase V is constant, and for any two positive frequencies w, w_0 one has $U(w) - U(w_0) = -\frac{2}{\pi}\int_{w_0}^{w} V(s)\,d\log s$. Thus, the approximation (1.8) is valid exactly in this case, except for the singularity at zero.
The log transfer function of an exponential filter $k(t) = \lambda\,\exp(-\lambda t)$, $t > 0$, is $\log\hat{k}(\omega) = -\log(1 - \frac{i\omega}{\lambda})$, so its real and imaginary parts are given by $U(\omega) = -\frac{1}{2}\,\log(1 + (\frac{\omega}{\lambda})^2)$ and $V(\omega) = \arctan(\frac{\omega}{\lambda})$, respectively.
In Figure 1 the function U and the approximations of the orders 1, 3, and 5 are plotted against the frequency w, for four different values of the scaling parameter b. Note that for $b = 25$, k is almost an delta - function.

Remark 2.3 The density $\sigma_k(r)/r$ is convex on $(0,1)$ for every $k \geq 0$, and is stretched to nearly a straight line, $\sigma_k(r)/r \approx r\,\sigma_k(1)$, for large k. This can be seen from the Taylor expansion of the logarithm,

$$\sigma_k(r) \;=\; \int_0^1 \frac{(\log\frac{1}{s})^k}{k!}\left(-\frac{1}{2}\,\log(1 - r^2 s^2)\right)\frac{ds}{s} \;=\; \sum_{m=1}^{\infty}\frac{1}{2}\,\frac{r^{2m}}{m}\int_0^1 \frac{(\log\frac{1}{s})^k}{k!}\,s^{2m}\,ds.$$

The claim follows since in the Taylor series $\sigma_k(r)/r = \sum_{m=1}^{\infty}\frac{r^{2m-1}}{2m(2m+1)^{k+1}}$ all coefficients are positive and such that for large k, it is only the first term which matters.
Likewise, the density $\tau_k(s) = \sigma_k(1/s)/s$ of the image measure of $\sigma_k(r)\,dr/r$ under the mapping $r \to s = 1/r$ also is convex on the interval $(1,\infty)$ and assumes the form $s^{-3}\,\sigma_k(1)$, $s > 1$, for large k.

3 The reverse relation

To establish a similar expansion for the reverse relation (1.4) one may proceed in much the same way as in the previous section. However, there are obstacles.

To be specific, let $U_k(w)$ denote the k-th derivative of U with respect to $\log w$. Proceeding as in the proof of Proposition 2.1 one can show that

$$
\begin{aligned}
w\int_0^\infty \frac{U(x)\,dx}{x^2-w^2} &= \int_0^1 (U(w/t)-U(wt))\,\frac{dt}{1-t^2} \\
&= \sum_{k=1}^{n-1} U_k(w)\,\frac{1}{k!}\int_0^1 \left((-\log t)^k-(\log t)^k\right)\frac{dt}{1-t^2} \;+\; R_n^U(w) \\
&= \sum_{k=1,\ k\ \mathrm{odd}}^{n-1} U_k(w)\,\zeta(k+1)\,(2-2^{-k}) \;+\; R_n^U(w)\,,
\end{aligned}
$$

where

$$
R_n^U(w) = \int_0^1 \left(\int_w^{w/t} K_n^U(w,x_1)\,\frac{dx_1}{x_1} - \int_w^{wt} K_n^U(w,x_1)\,\frac{dx_1}{x_1}\right)\frac{dt}{1-t^2},
$$

and K_n^U is defined as in (2.10), except that V is replaced by U.
The problem alluded to above now becomes clear. Due to the change of the "weigthing measure" from $\frac{t\,dt}{1-t^2}$ to $\frac{dt}{1-t^2}$ the coefficients in the expansion do not decrease to zero but to the constant 2. Therefore the approximation

$$
V(w) \approx -\frac{2}{\pi}\sum_{k=1,\ k\ \mathrm{odd}}^{n-1} U_k(w)\,\zeta(k+1)\,(2-2^{-k})
$$

cannot be expected to be effective.
This problem can be overcome by starting from an alternative logarithmic Kramers–Kronig relation, to be derived in the appendix, namely

$$
V(w) - w\,V'(0) \;=\; -\frac{2w^3}{\pi}\int_0^\infty \frac{U(x)-U(0)}{x^2\,(x^2-w^2)}\,dx. \tag{3.1}
$$

If we then define a function Q by $Q(w) = (U(w)-U(0))/w$ and replace V in the last section by Q then everything goes through in exactly the same fashion. That is, up to a change of sign we obtain the exact analogue of Proposition 2.1 – with assumption (A_2) instead of (A_1) – if we replace V by Q and $U(w)-U(0)$ on the left-hand side of (2.1) by $V(w)-w\,V'(0)$: we then have

$$
\frac{\pi}{2}\,(V(w)-w\,V'(0)) \;=\; \int_0^w Q(x)\,d\log x \;-\; \sum_{k=1,\,k\ odd}^{n-1} Q_k(w)\,\zeta(k+1)\,2^{-k} \;-\; R_n^Q(w)\,. \tag{3.2}
$$

Remark 3.1 Notice that the imaginary part V of the log transfer function cannot be recovered entirely from the real part U through (3.1), but only up to a linear function of w. A similar remark also applies to (1.5), of course, which gives U in terms of V only up to a constant.

4 Appendix

The validity of the logarithmic Kramers–Kronig relations can be established by means of the following lemma.

Lemma 4.1 *Suppose the function F is analytic in $\mathbf{C}_+$, continuous on the closed half plane $\overline{\mathbf{C}}_+$, and satisfies the conditions*

(C1) $\lim_{R\to\infty} R^{-1} \int_0^\pi |F(Re^{i\phi})|\, d\phi \;=\; 0$;

(C2) $\int_{-\infty}^{\infty} \frac{|F(x)|}{1+x^2}\, dx \;<\; \infty$.

Then for every $w > 0$ we have

$$PV\!\int_{-\infty}^{\infty} \frac{F(x)\, dx}{x^2 - w^2} \;=\; \frac{\pi i}{2w}\,(F(w) - F(-w)). \tag{4.1}$$

PROOF (sketched). Let $w > 0$ be fixed. By Cauchy's theorem, $\int_C \frac{F(z)\,dz}{z^2-w^2} = 0$ for the contour C consisting of the (big) semi-circle in $\overline{\mathbf{C}}_+$ around the origin with radius $R > w$ and the interval $[-R, R]$, with the proviso that the points $\pm w$ are circumvented on small semi-circles in $\overline{\mathbf{C}}_+$ with radius r, say, as usual. Letting R tend to infinity makes the integral along the big semi-circle vanish, by (C1). As r tends to zero the small semi-circles contribute the amounts $-\pi i\,\frac{F(\pm w)}{\pm 2w}$, respectively, by the assumed continuity of F on $\overline{\mathbf{C}}_+$. Finally, consider the integral over the remaining part of C, the union of intervals $J(R,r) = [-R, -w-r] \cup [-w+r, w-r] \cup [w+r, R]$. As R tends to infinity the corresponding integral tends to $\int_{J(\infty,r)} \frac{F(x)\,dx}{x^2-w^2}$, by (C2). Letting then r tend to zero makes this last integral converge to the Cauchy principal value. Since the sum of all integrals equals zero the lemma follows.

Corollary 4.1 *Let F_1, F_2 denote the real and the imaginary part of F, $F = F_1 + iF_2$. If F_1 is even and F_2 is odd (as functions on the real line) then*

$$F_2(w) = -\frac{2w}{\pi}\; PV\!\int_0^{\infty} \frac{F_1(x)\, dx}{x^2 - w^2} \qquad (w > 0). \tag{4.2}$$

Conversely, if F_1 is odd and F_2 is even then

$$F_1(w) = \frac{2w}{\pi}\; PV\!\int_0^{\infty} \frac{F_2(x)\, dx}{x^2 - w^2} \qquad (w > 0). \tag{4.3}$$

PROOF. This is easily proved by equating the real and imaginary parts and observing that $PV\!\int_{-\infty}^{\infty} \frac{g(x)\,dx}{x^2-w^2}$ vanishes if g is odd.

Corollary 4.2 *The relations (1.4), (1.5), and (3.1) are valid under the assumptions (A_0), (A_1), and (A_2), respectively.*

PROOF. This follows from Corollary 4.1 by suitable choices of F. In the first case, take $F(\omega) = \log \hat{k}(\omega)$ (so that $F_1(x) = U(x)$ is even and $F_2(x) = V(x)$ is odd). Relation (1.5) may be obtained by choosing $F(\omega) = (\log \hat{k}(\omega) - \log \hat{k}(0))/\omega$ (so that $F_1(x) = (U(x) - U(0))/x$ is odd and $F_2(x) = V(x)/x$ is even). The modification (3.1) of the relation (1.4), finally, is obtained by choosing $F(\omega) = (\log \hat{k}(\omega) - \log \hat{k}(0) - w\,\hat{k}'(0)/\hat{k}(0))/\omega^2$ (so that $F_1(x) = (U(x) - U(0) - x\,U'(0))/x^2 = (U(x) - U(0))/x^2$ is even and $F_2(x) = (V(x) - V(0) - x\,V'(0))/x^2 = (V(x) - x\,V'(0))/x^2$ is odd).
It remains to verify that in each case the assumptions (A_q) imply the conditions of Lemma

4.1. Analyticity of F in $\mathbf{C}_+$ is clear, and (C1), (C2) immediately follow from (1.6), (1.7) and the respective definition of F. Continuity on $\overline{\mathbf{C}}_+$ is a consequence of the condition $\int_0^\infty t^q\,|k(t)|\,dt < \infty$, which entails the existence and continuity of the q-th derivative of $\hat{k}$, and hence, since $\hat{k}$ is zero-free on $\overline{\mathbf{C}}_+$, the continuity of F on $\overline{\mathbf{C}}_+$. This completes the proof.

References

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